Algorithm Problem Solving (APS):
Sorting

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UC San Diego SPIS 2019

Introduction to Sorting

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    - This is for sorting in *ascending* order
    - Just change “less than” to “greater than” for *descending* order

Introduction to Sorting

- How do we sort *n* items?

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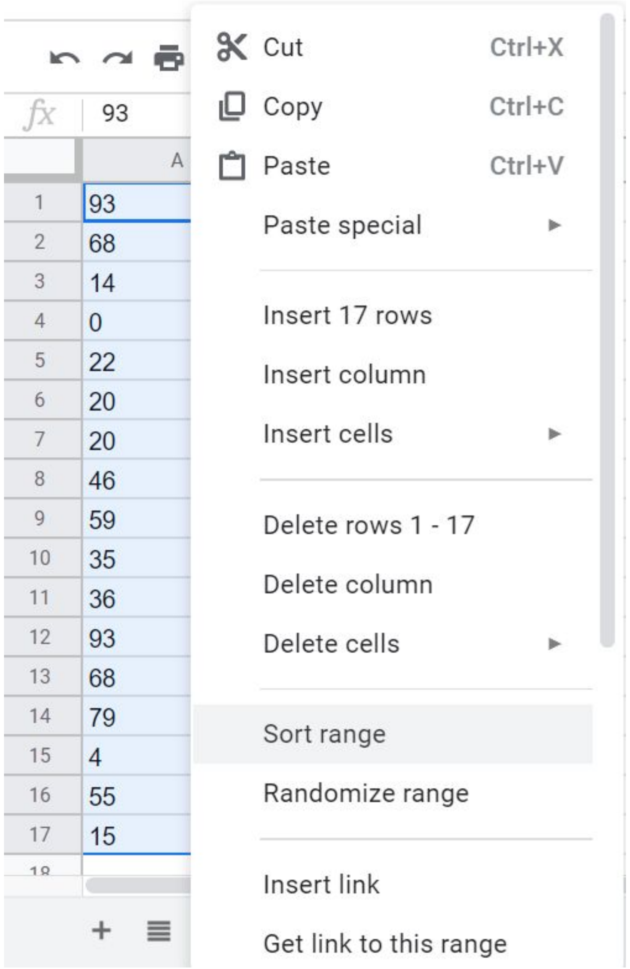

	A
1	93
2	68
3	14
4	0
5	22
6	20
7	20
8	46
9	59
10	35
11	36
12	93
13	68
14	79
15	4
16	55
17	15

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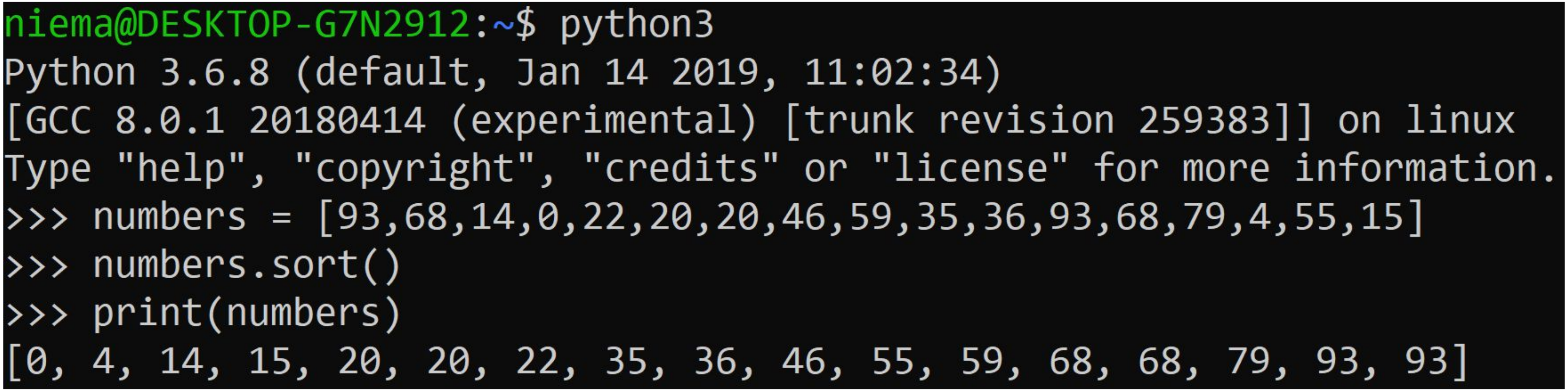

```
niema@DESKTOP-G7N2912:~$ python3
Python 3.6.8 (default, Jan 14 2019, 11:02:34)
[GCC 8.0.1 20180414 (experimental) [trunk revision 259383]] on linux
Type "help", "copyright", "credits" or "license" for more information.
>>> numbers = [93,68,14,0,22,20,20,46,59,35,36,93,68,79,4,55,15]
>>> numbers.sort()
>>> print(numbers)
[0, 4, 14, 15, 20, 20, 22, 35, 36, 46, 55, 59, 68, 68, 79, 93, 93]
```

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- But first, let's discuss time complexity using Big-O notation

Time Complexity

Describing an Algorithm

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    - Human Time (e.g. seconds)
    - Computer Time (e.g. clock cycles)

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  - Can we describe an algorithm independently of implementation?

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  - But with what input data?

The Best, the Worst, and the Average

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- People typically mainly care about the worst case
    - “Your package will arrive in around 1 to 100 days”

Big-O, Big-Ω, and Big-Θ

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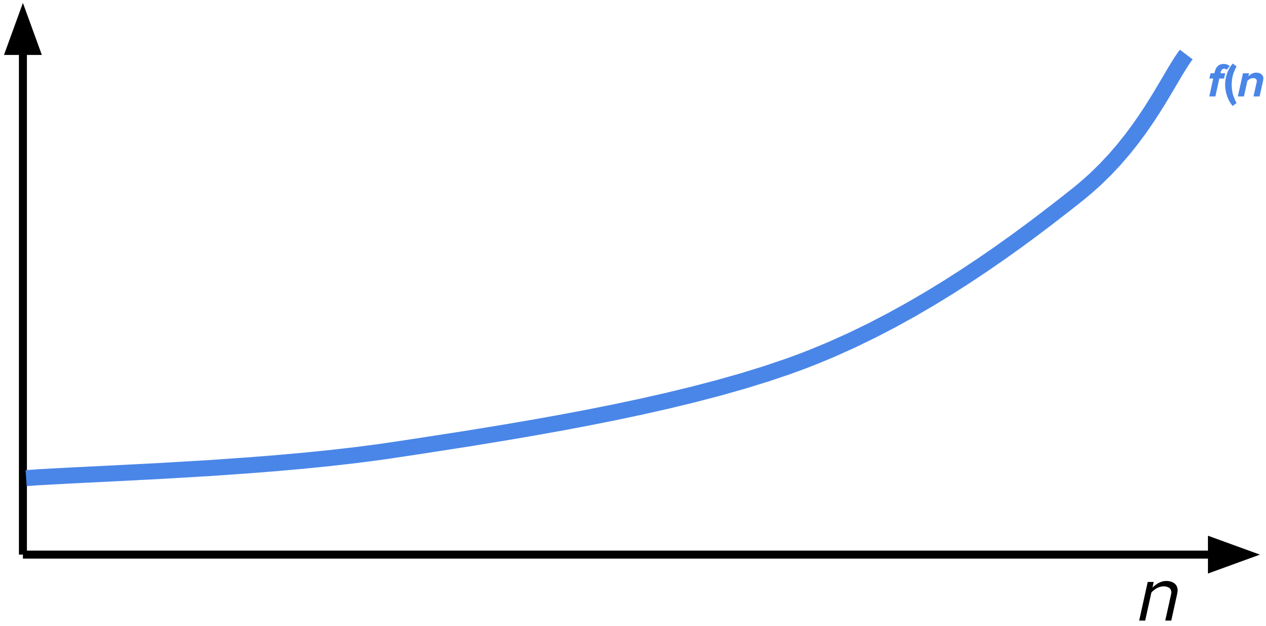
Example: Big-O, Big-Ω, and Big-ϴ

- Number of Operations = $f(n) = 2n^2 + 3n + 1$



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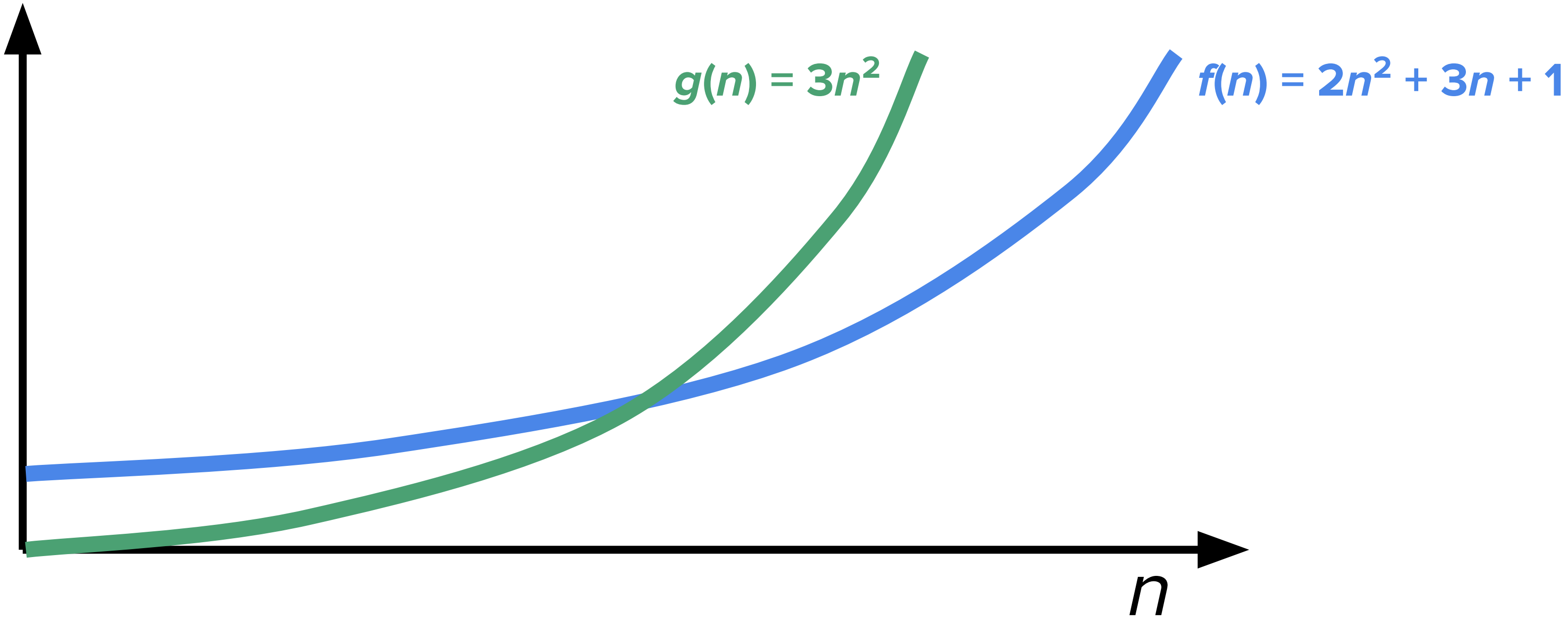

Ex

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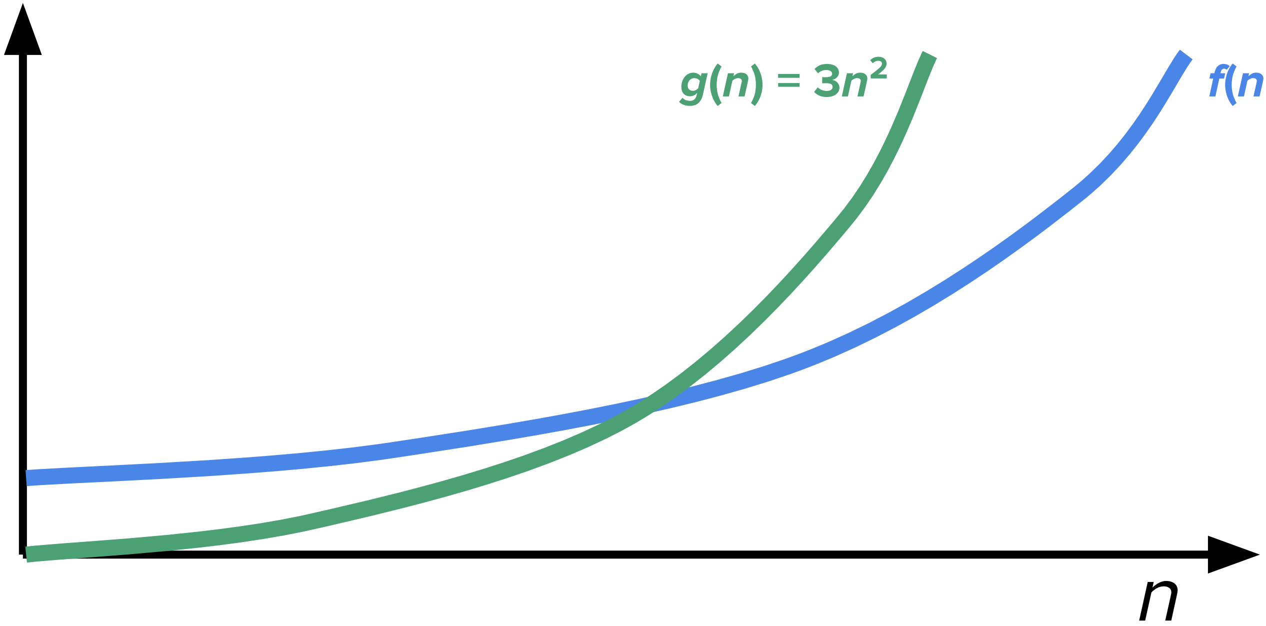

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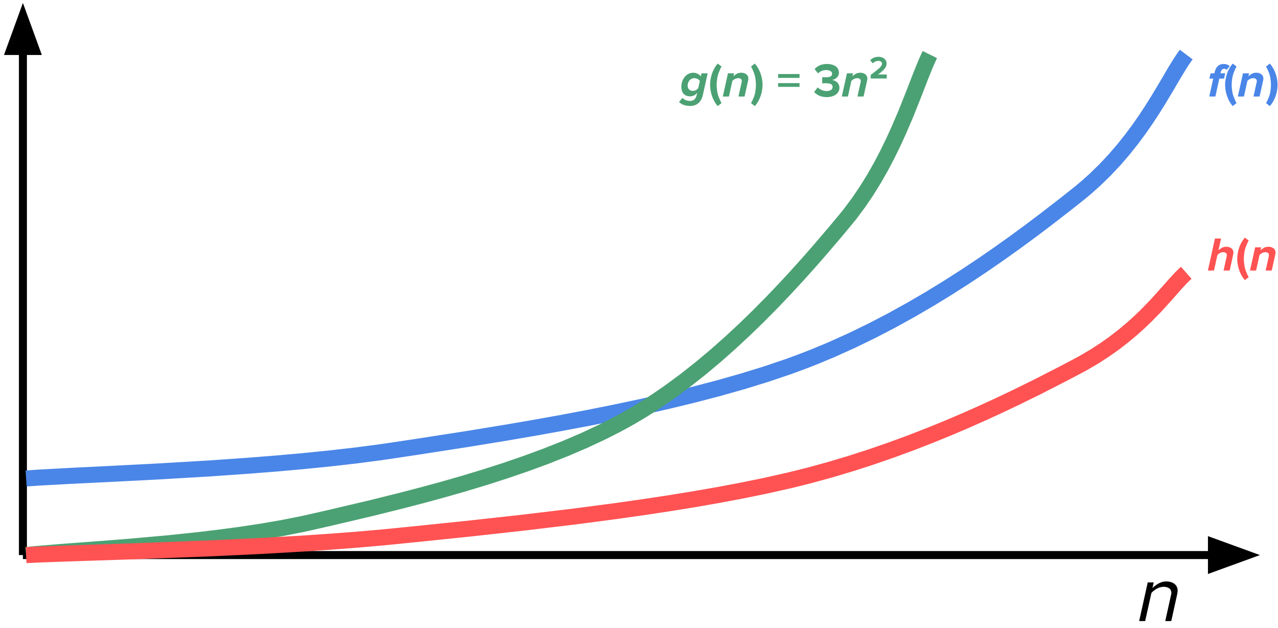

Ex

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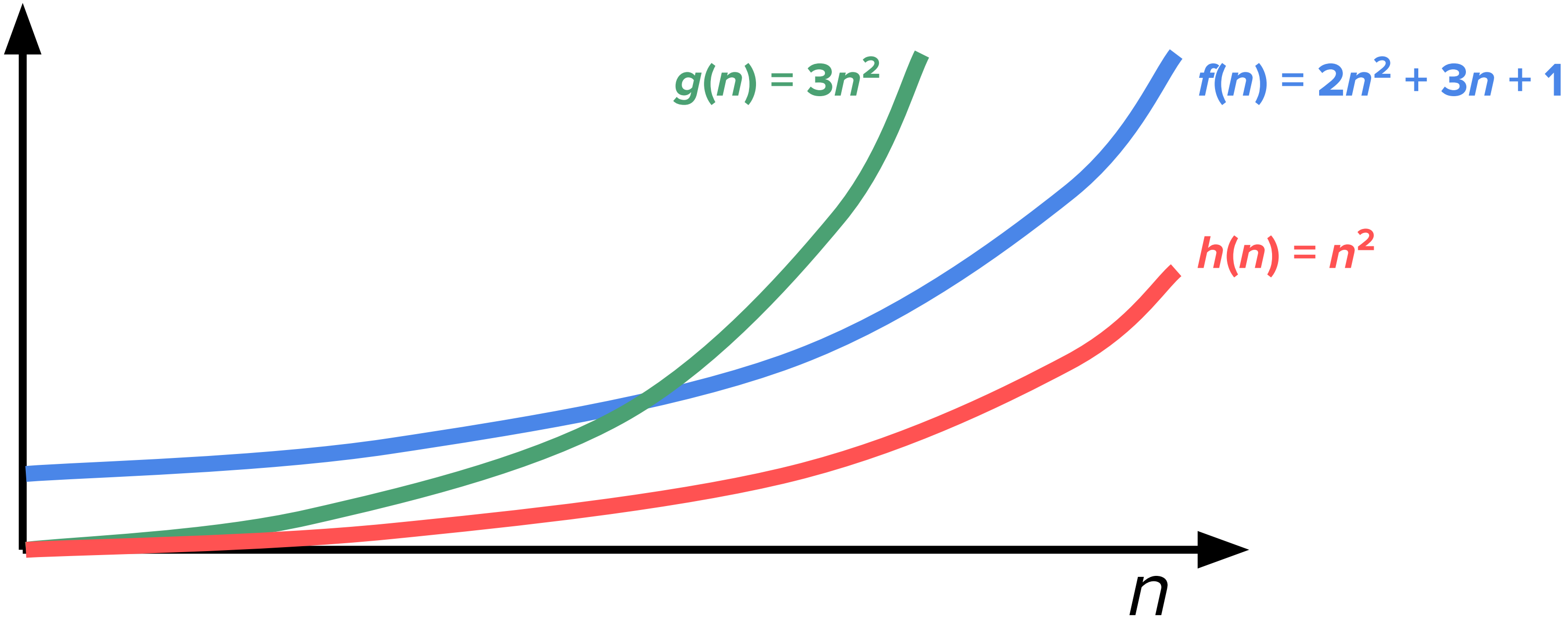

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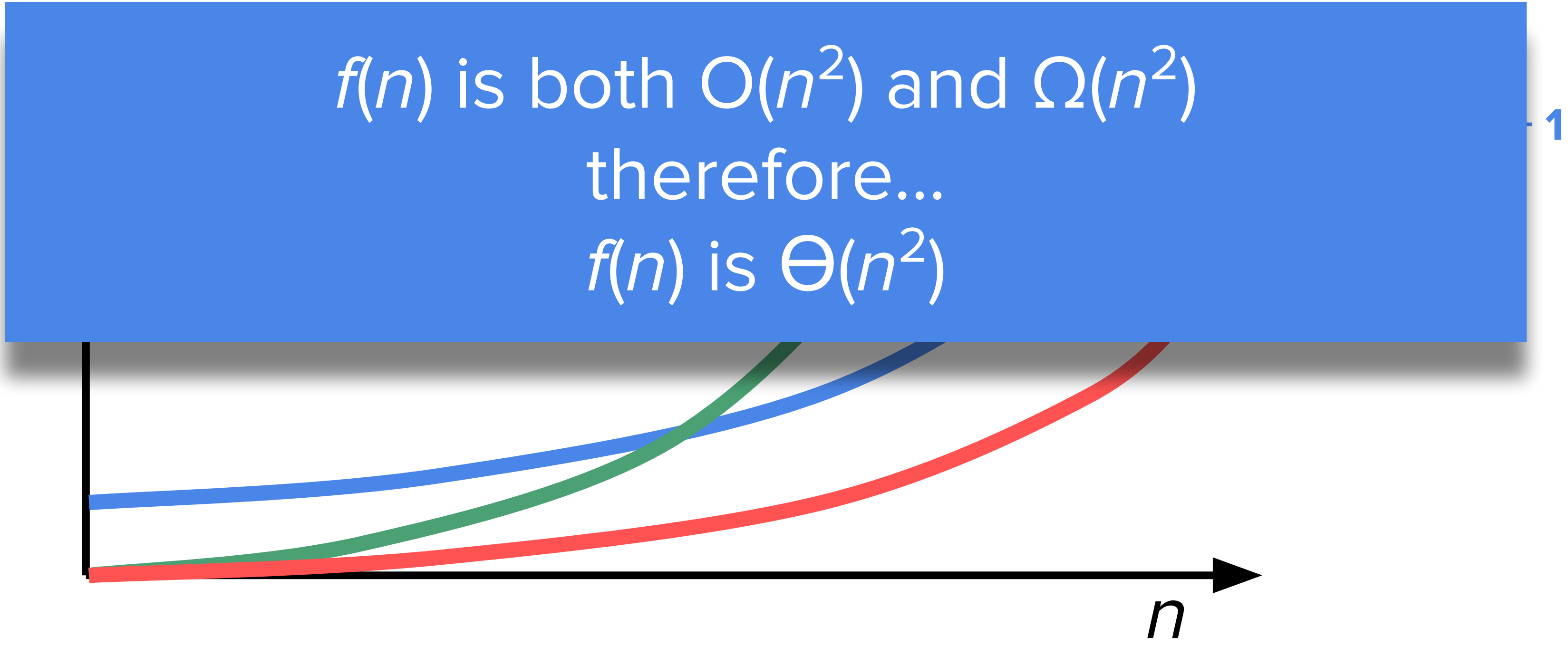

$f(n)$ is both $O(n^2)$ and $\Omega(n^2)$

therefore...

$f(n)$ is $\Theta(n^2)$

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  - $5n \log n + \cancel{2n} + \cancel{27} \rightarrow \mathbf{5n \log n}$
  - $\cancel{5}n \log n \rightarrow n \log n \rightarrow \mathbf{O(n \log n)}$

Selection Sort

Selection Sort

```
Algorithm selection_sort(X):
    output ← empty list
    Repeat |X| times:
        y ← smallest item in X
        Remove y from X
        Add y to output
    Return output
```

Selection Sort

7	25	0	42	-9

Selection Sort

7	25	0	42	**-9**

Selection Sort

7	25	0	42	**-9**

-9				

Selection Sort

7	25	0	42	~~-9~~

-9				

Selection Sort

7	25	**0**	42	~~-9~~

-9				

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7	25	**0**	42	~~-9~~

-9	0			

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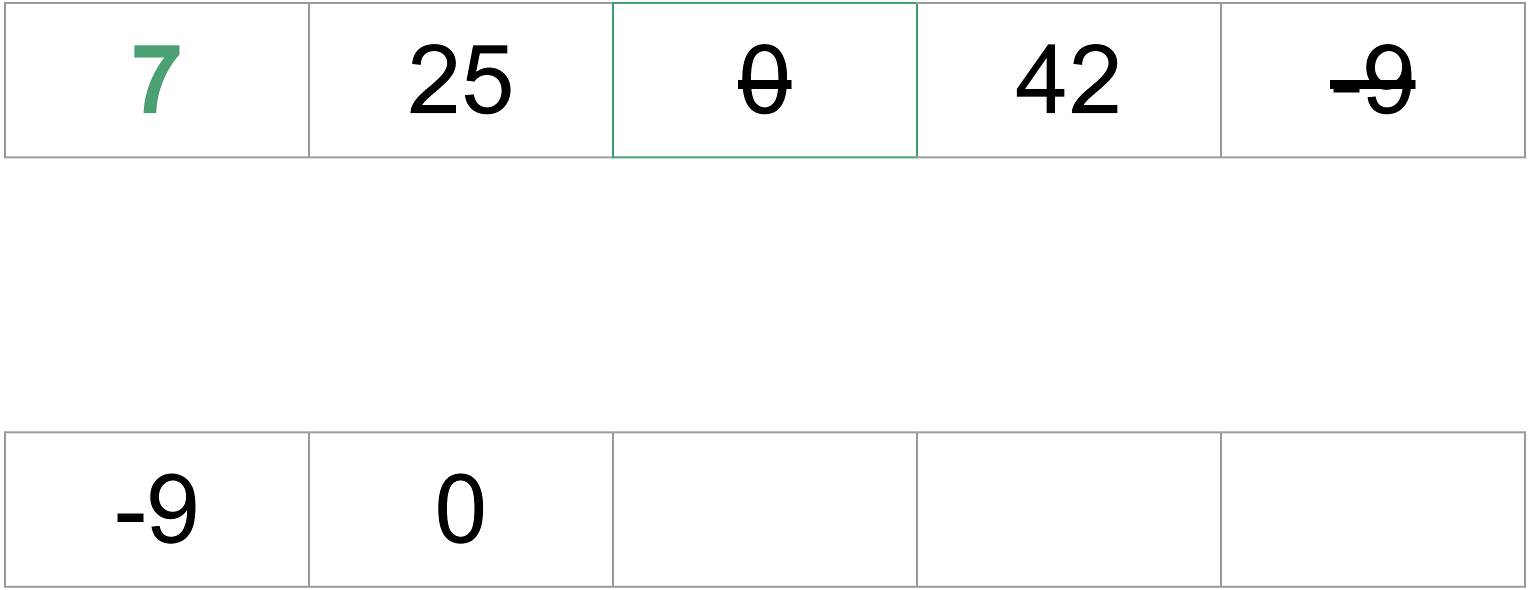


7	25	~~0~~	42	~~-9~~

-9	0			

Selection Sort

7	25	~~0~~	42	~~-9~~

-9	0	7		

Selection Sort

~~7~~	25	~~0~~	42	~~-9~~

-9	0	7		

Selection Sort

~~7~~	**25**	~~0~~	42	~~-9~~

-9	0	7		

Selection Sort

~~7~~	**25**	~~0~~	42	~~-9~~

-9	0	7	25	

Selection Sort

~~7~~	~~25~~	~~0~~	42	~~-9~~

-9	0	7	25	

Selection Sort

~~7~~	~~25~~	~~0~~	**42**	~~-9~~

-9	0	7	25	

Selection Sort

~~7~~	~~25~~	~~0~~	**42**	~~-9~~

-9	0	7	25	42

Selection Sort

~~7~~	~~25~~	~~0~~	~~42~~	~~-9~~

-9	0	7	25	42

Selection Sort

~~7~~	~~25~~	~~0~~	~~42~~	~~-9~~

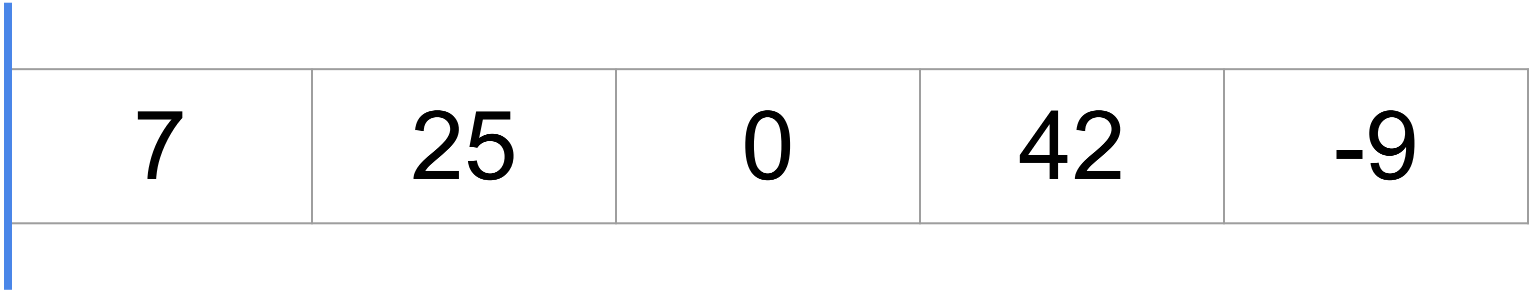
Can we do it in-place?

-9	0	7	25	42

Selection Sort (In-Place)

7	25	0	42	-9

Selection Sort (In-Place)



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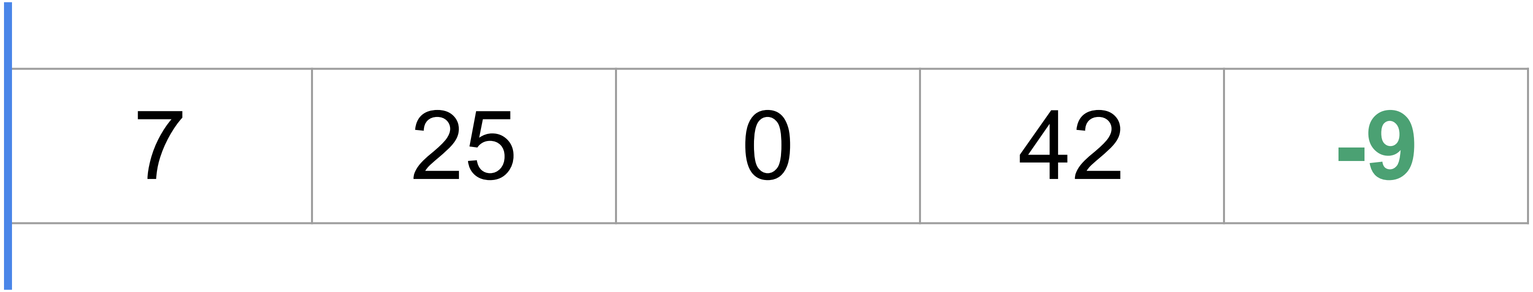

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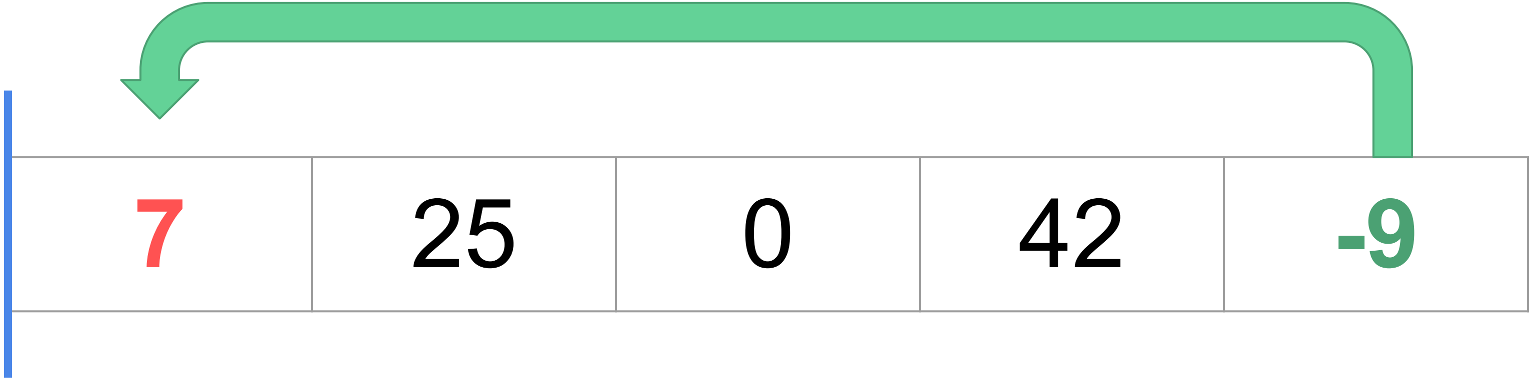

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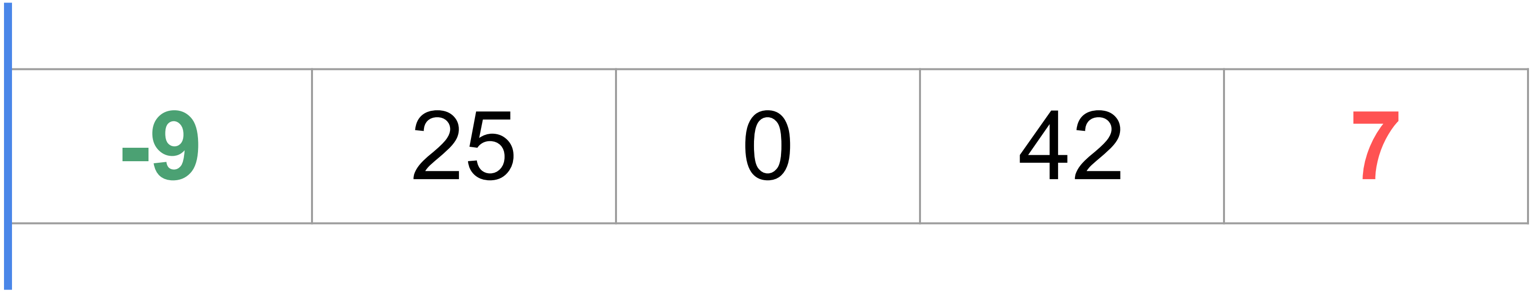
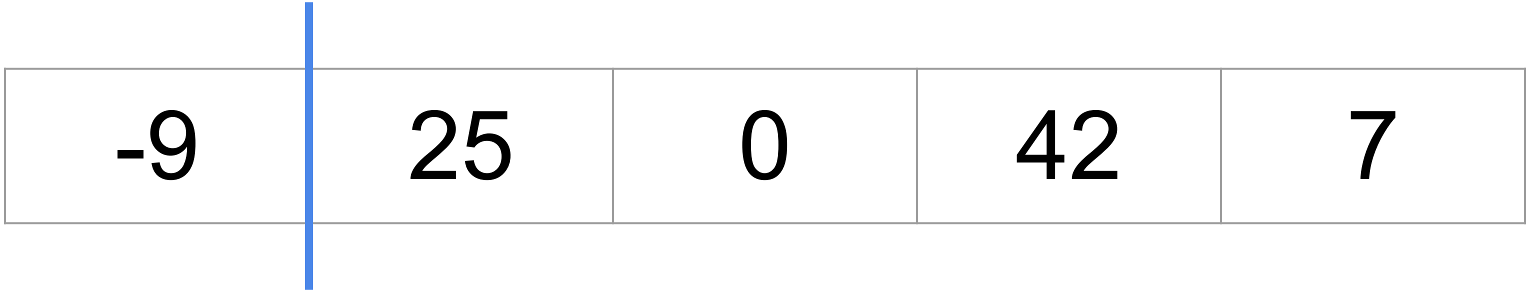


-9	25	0	42	7

Selection Sort (In-Place)



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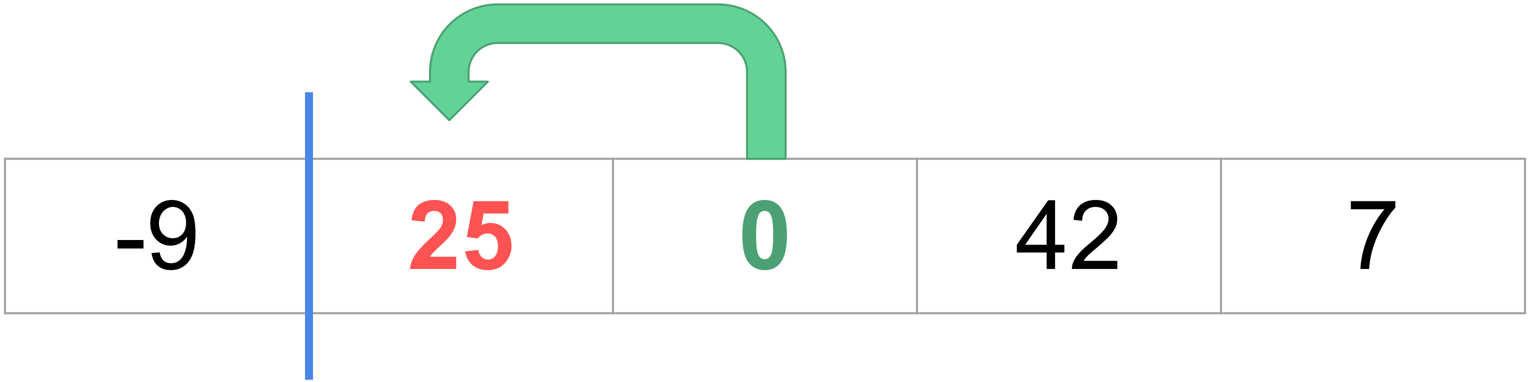

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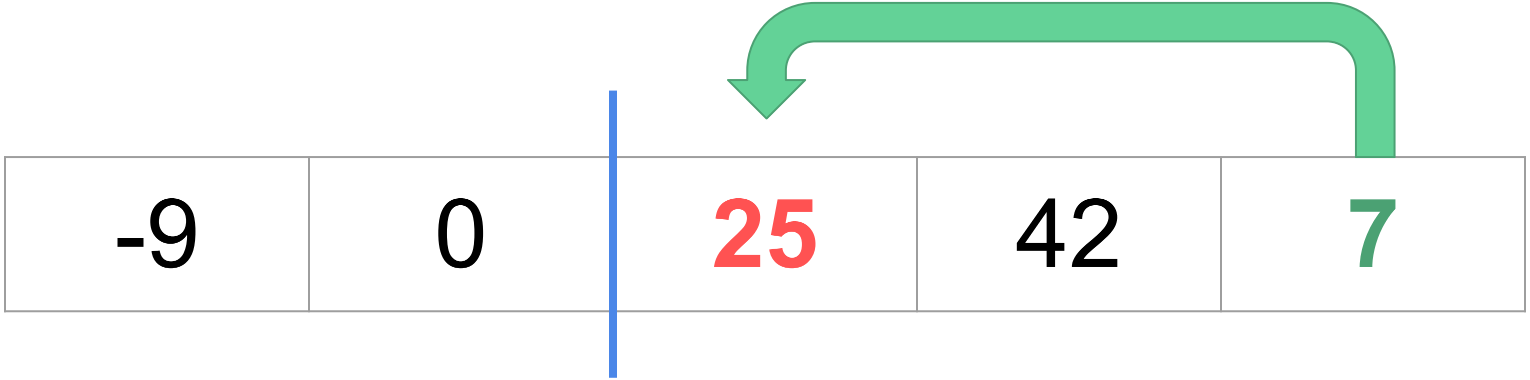

-9	0	25	42	7

Selection Sort (In-Place)

-9	0	7	42	25

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-9	0	7	42	25

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-9	0	7	42	**25**

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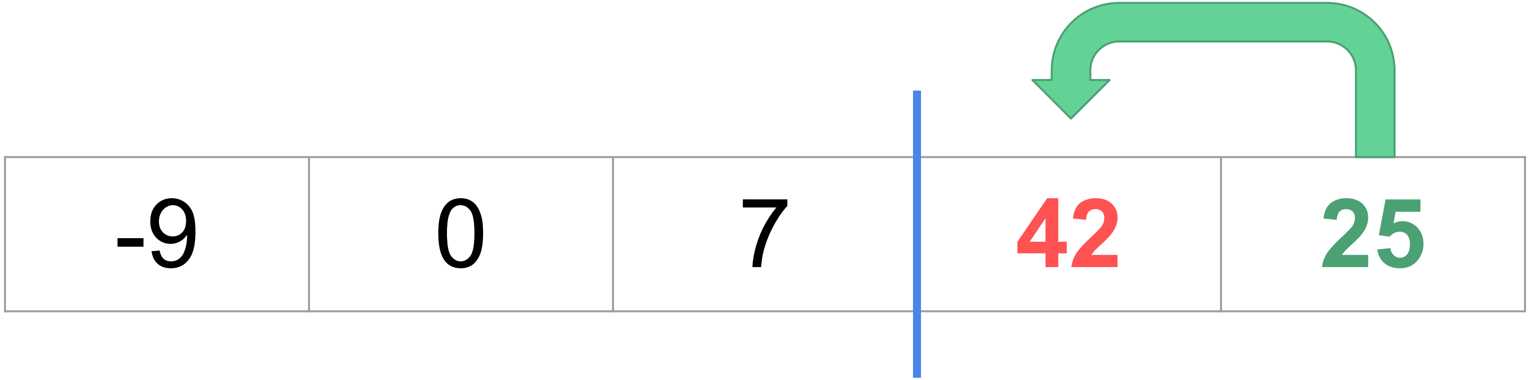


-9	0	7	42	25

Selection Sort (In-Place)

-9	0	7	**25**	**42**

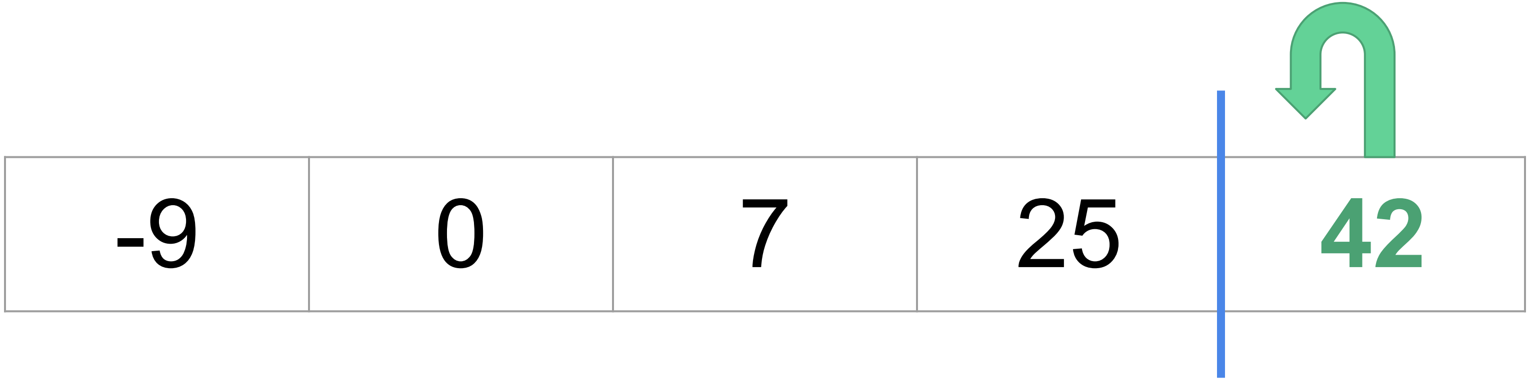
Selection Sort (In-Place)

-9	0	7	25	42

Selection Sort (In-Place)

-9	0	7	25	**42**

Selection Sort (In-Place)



-9	0	7	25	**42**

Selection Sort (In-Place)

-9	0	7	25	42

Selection Sort (In-Place)

What type of algorithm is this?

-9	0	7	25	42

Selection Sort: Worst-Case Time Complexity

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- For each of our *n* iterations:

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  - $n(n+1)/2 = n^2 + n \rightarrow \mathbf{O(n^2)}$

Selection Sort: Worst-Case Time Complexity

- For each of our n iterations:
    - Find the smallest remaining item

Can we do better?

- Total number of operations = $n + (n-1) + (n-2) + \ldots + 3 + 2 + 1$
    - This is the sum of the integers from 1 to n, which is $n(n+1)/2$
    - $n(n+1)/2 = n^2 + n$ ➔ **O(n^2)**

Merge Sort

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```
Algorithm merge_sort(X):
    If |X| only has 1 item:
        Return |X|
    left ← merge_sort(left half of X)
    right ← merge_sort(right half of X)
    Return the result of merging left and right
```

Merge Sort

-9	0	7	25	42	5	-2	12

Merge Sort

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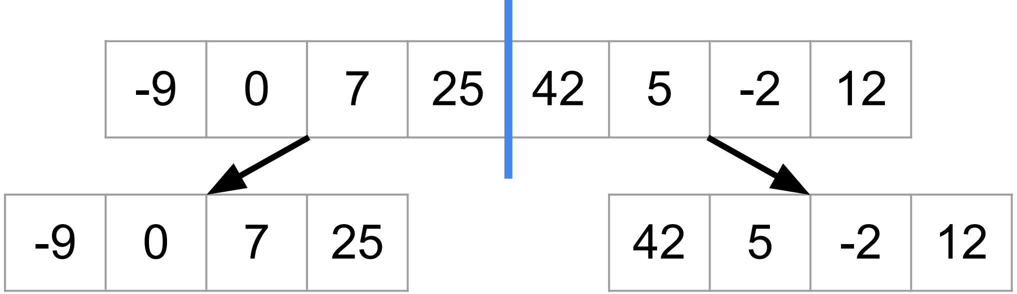
Merge Sort



-9	0	7	25	42	5	-2	12

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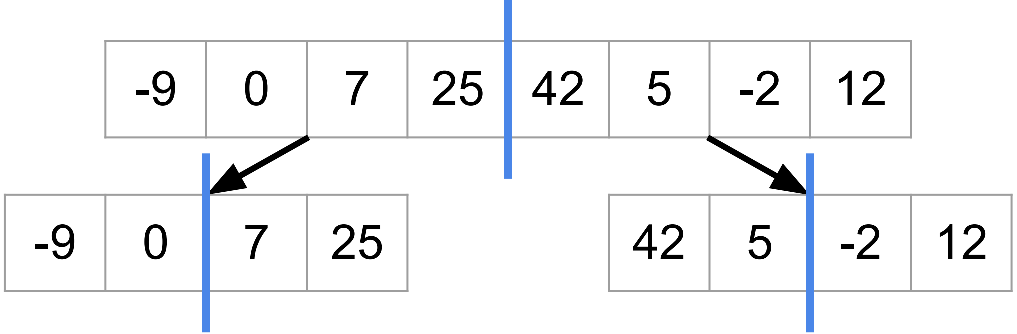
42	5	-2	12

Merge Sort



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-9	0	7	25

42	5	-2	12

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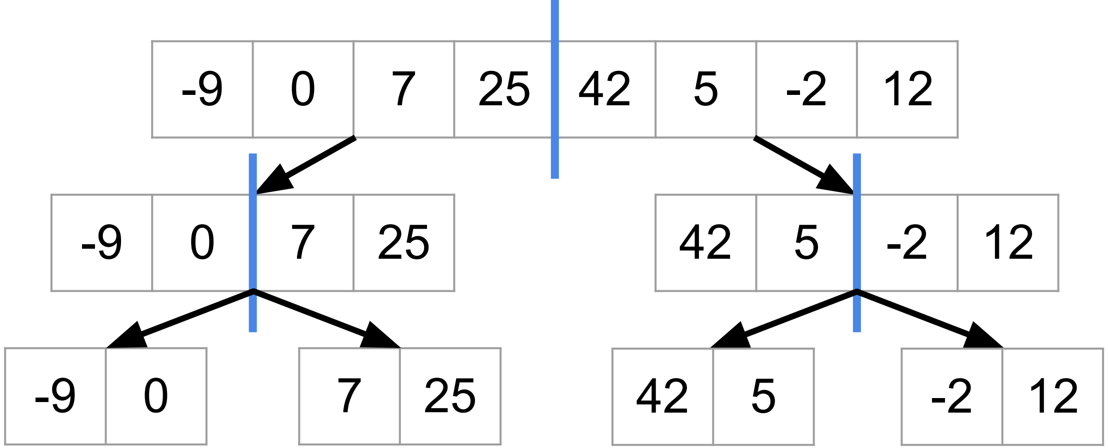

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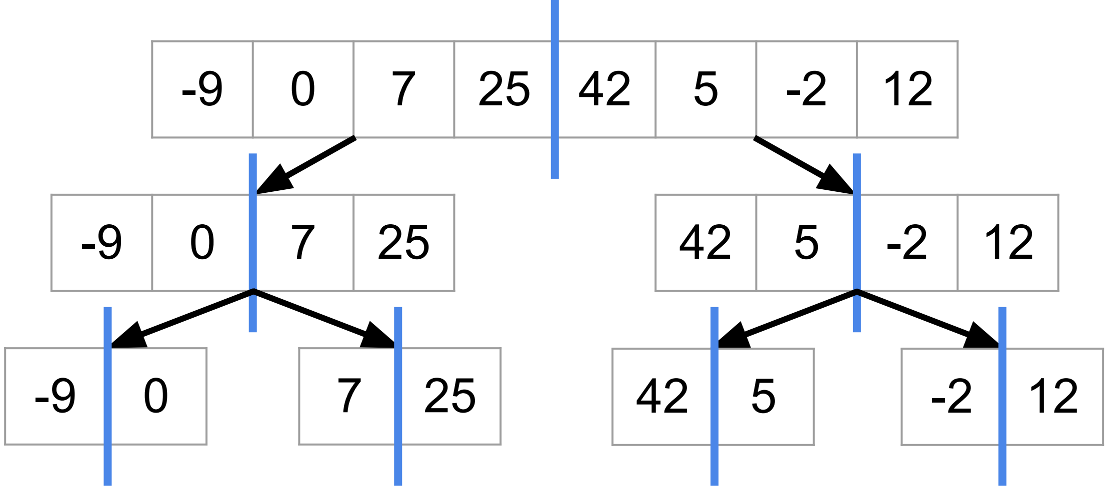

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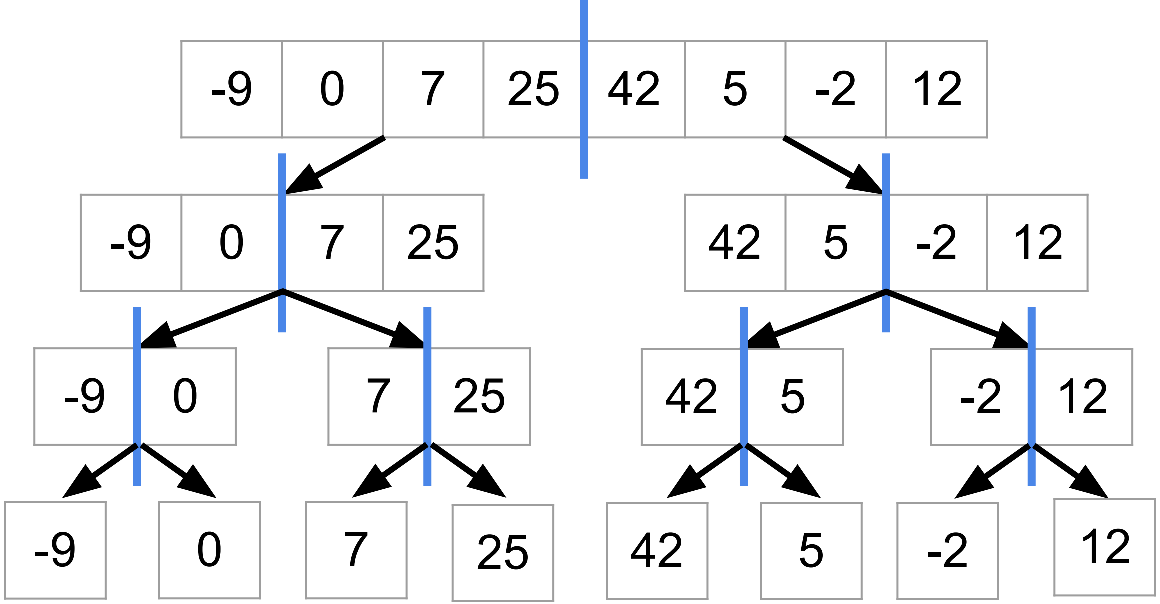

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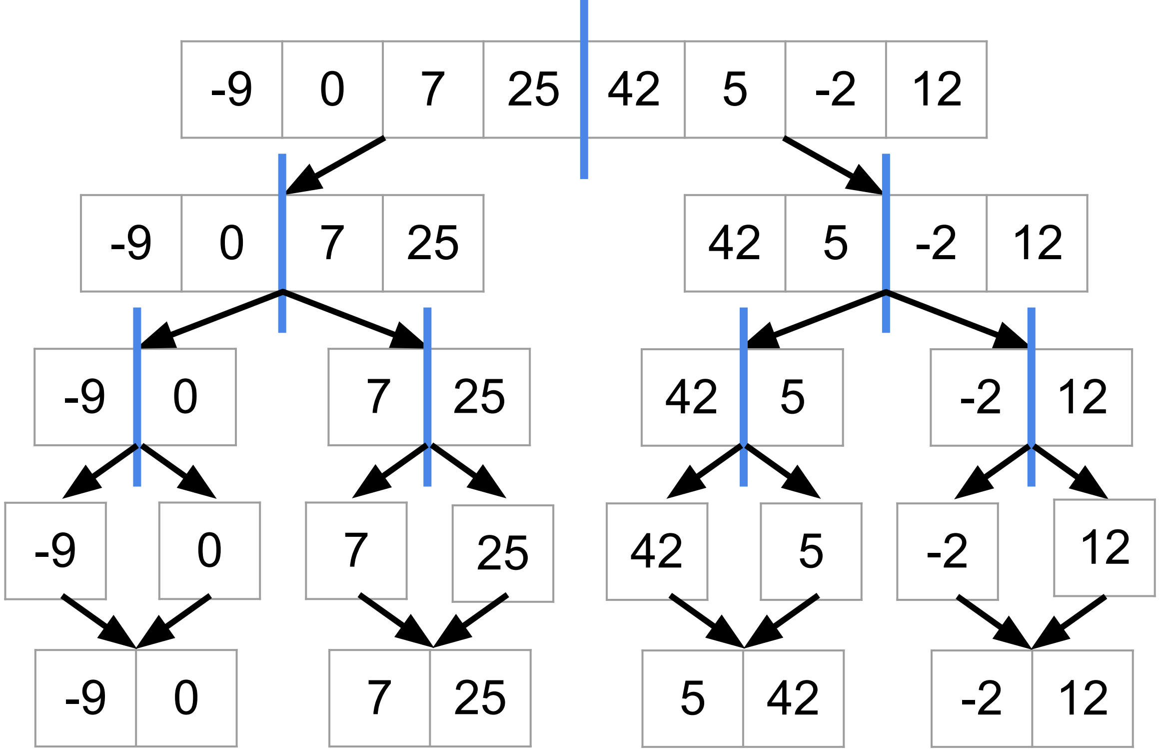
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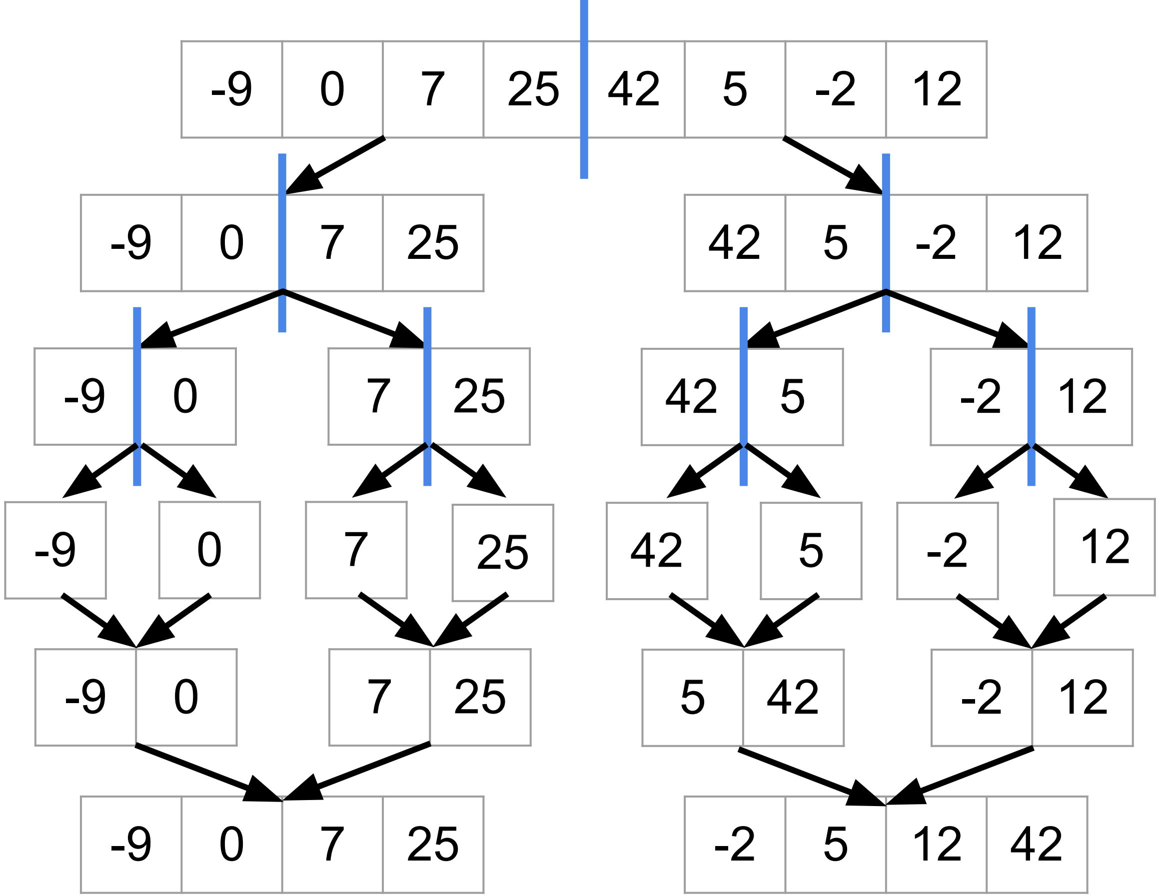
-9 0 7 25 42 5 -2 12
-9 0 7 25
42 5 -2 12
-9 0
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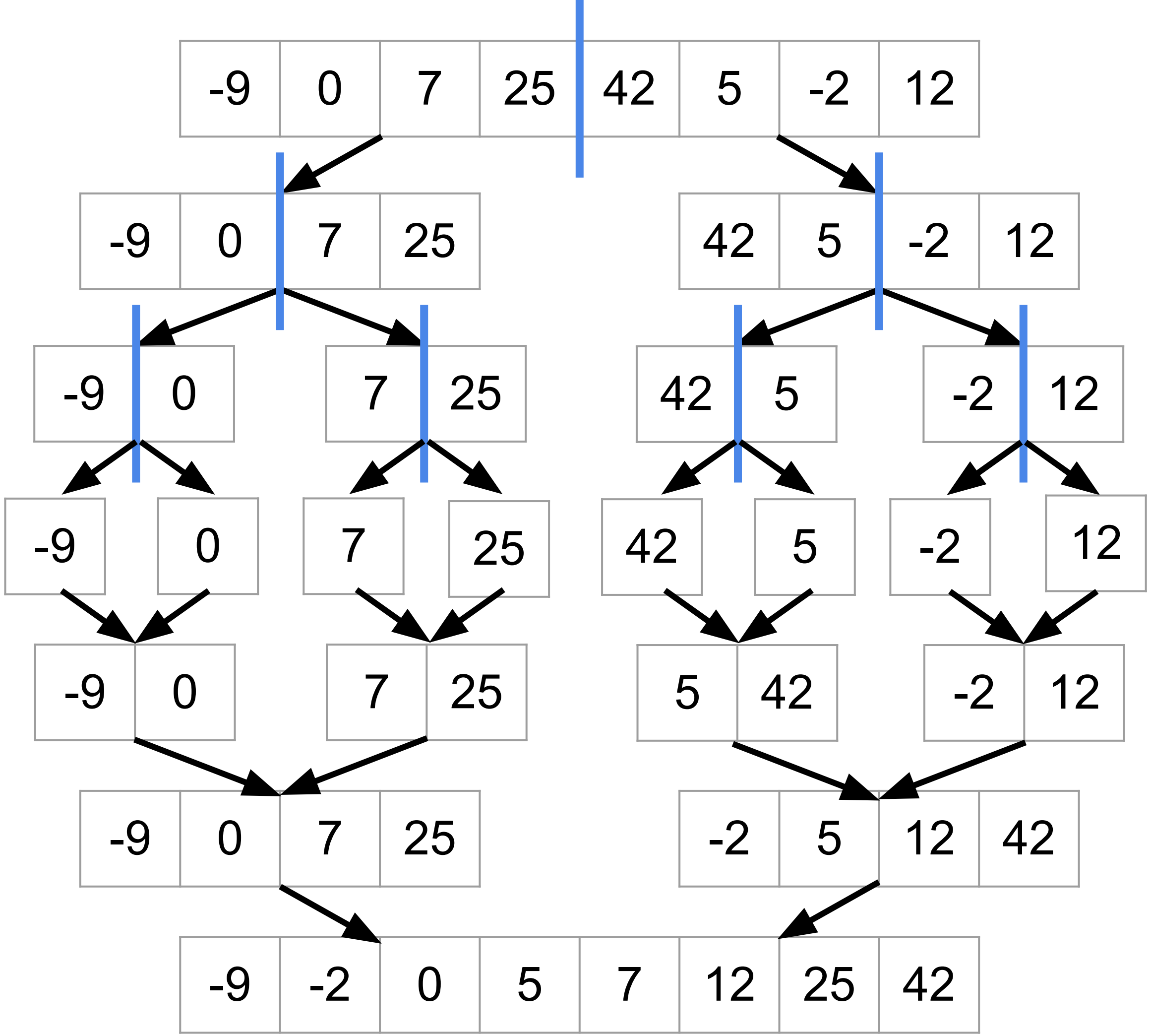
42
5
-2
12
-9 0
7 25
5 42
-2 12
-9 0 7 25
-2 5 12 42

Merge Sort


-9 0 7 25 42 5 -2 12
-9 0 7 25
42 5 -2 12
-9 0
7 25
42 5
-2 12
-9
0
7
25
42
5
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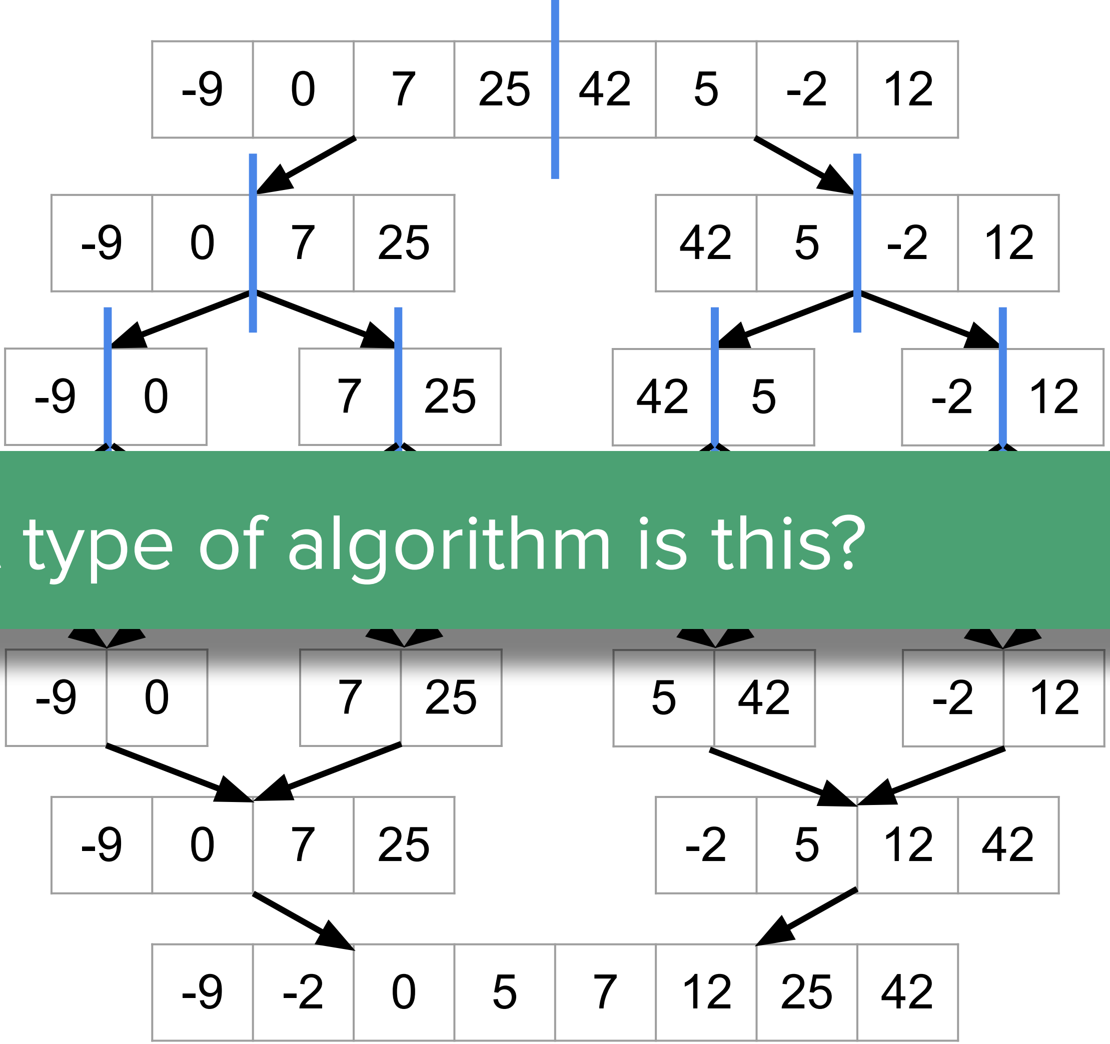
-9 -2 0 5 7 12 25 42

Merge Sort

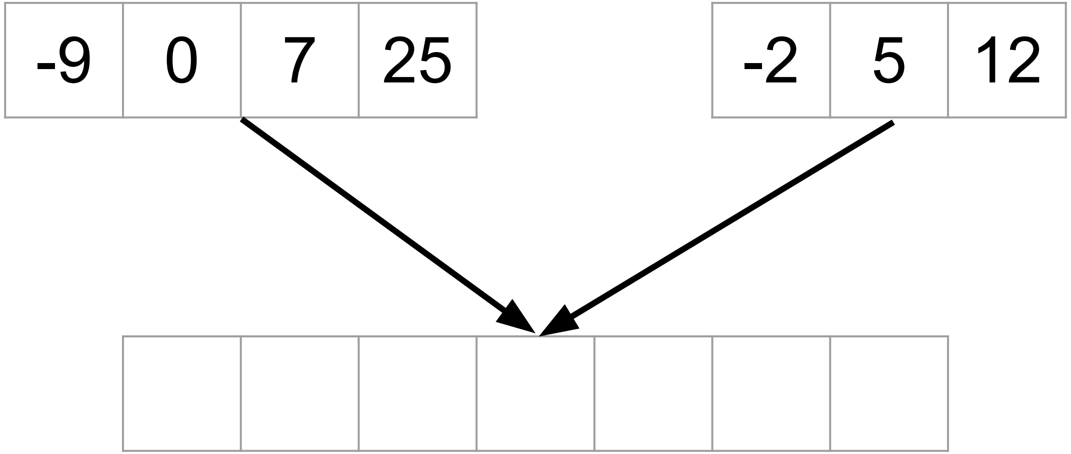


What type of algorithm is this?

Merging Two Sorted Lists

```
Algorithm merge(X,Y):
    output ← empty list;   i,j ← 0
    While i < |X| and j < |Y|:
        If X[i] < Y[j]:
            Add X[i] to output and increment i
        Else:
            Add Y[j] to output and increment j
    Add remaining items to output
    Return output
```

Merging Two Sorted Lists



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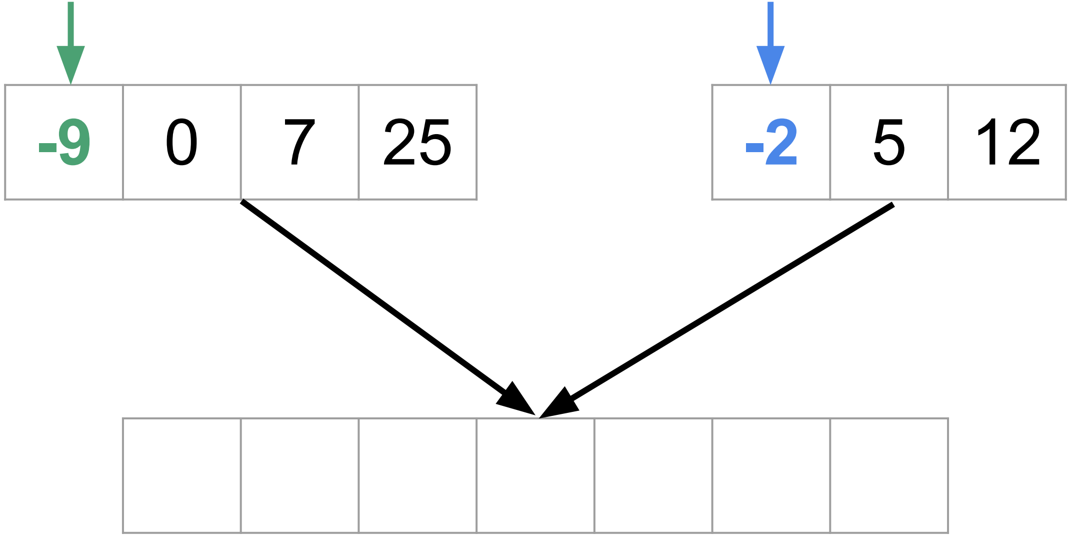

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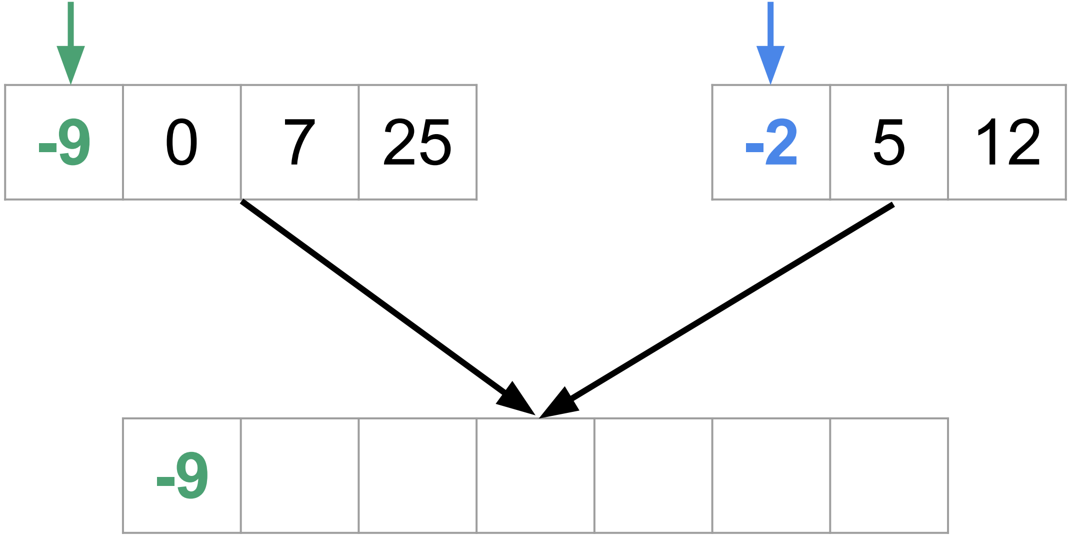

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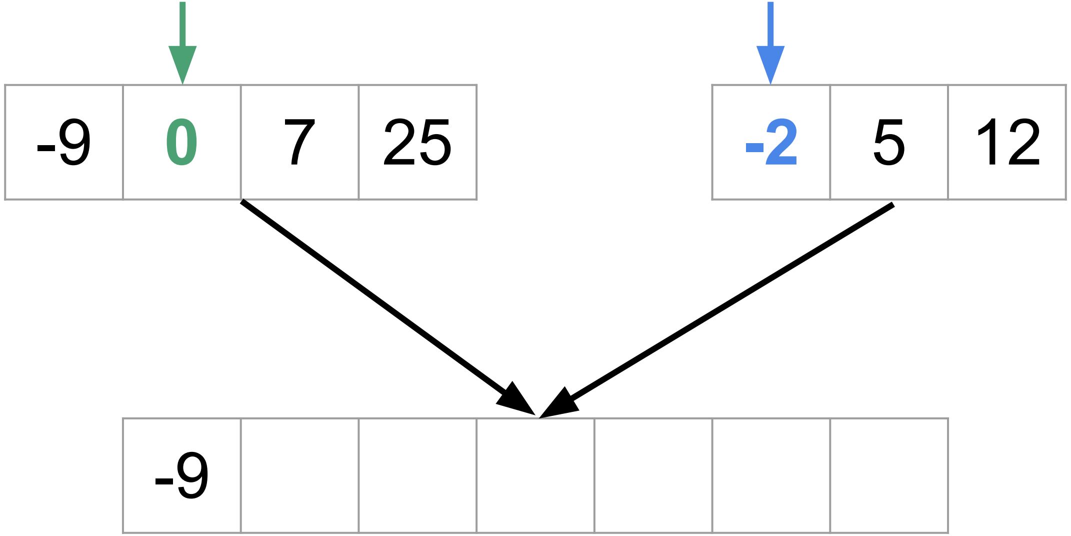

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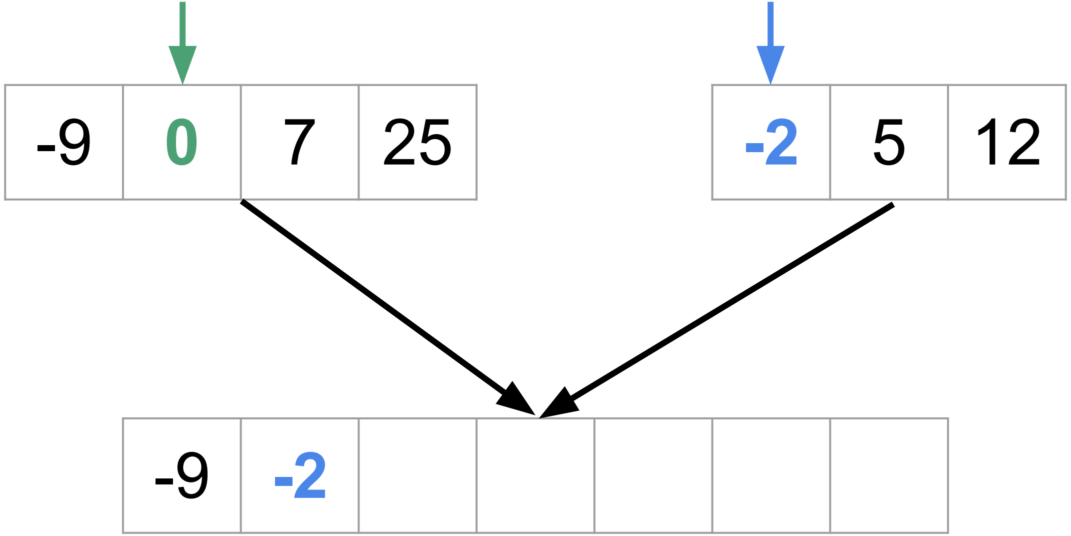

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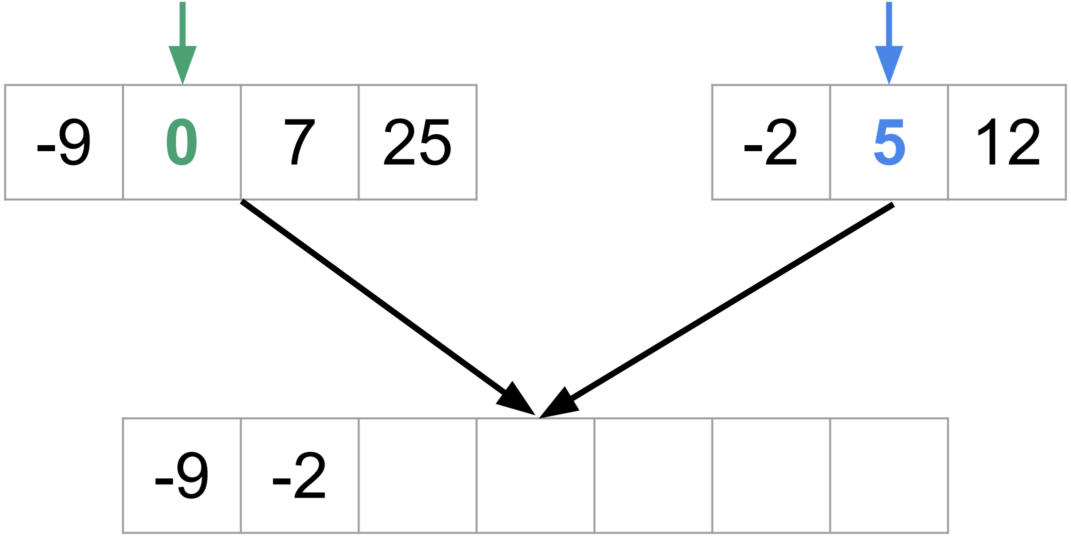
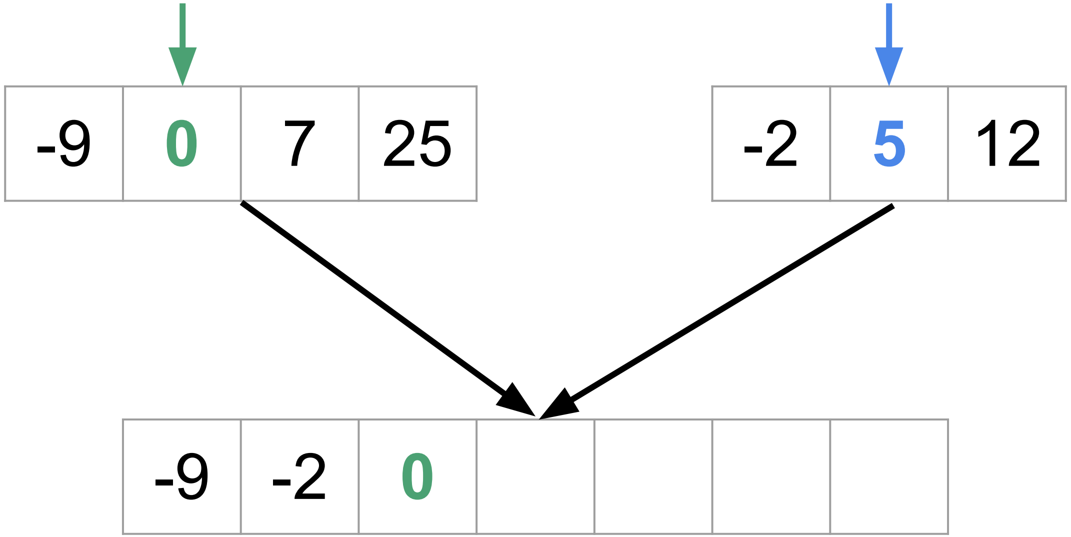


-9	**0**	7	25

-2	**5**	12

-9	-2					

Merging Two Sorted Lists



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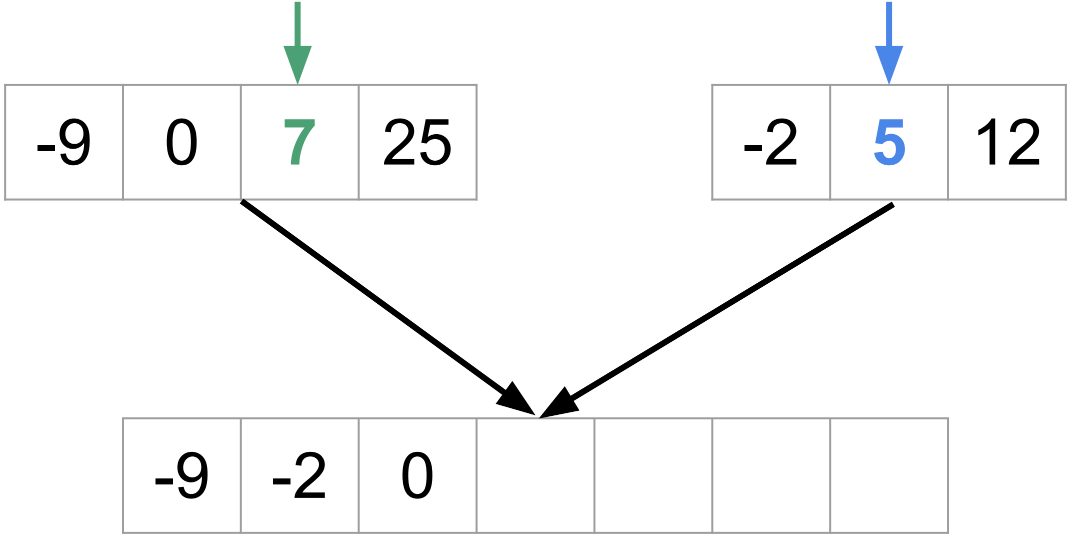

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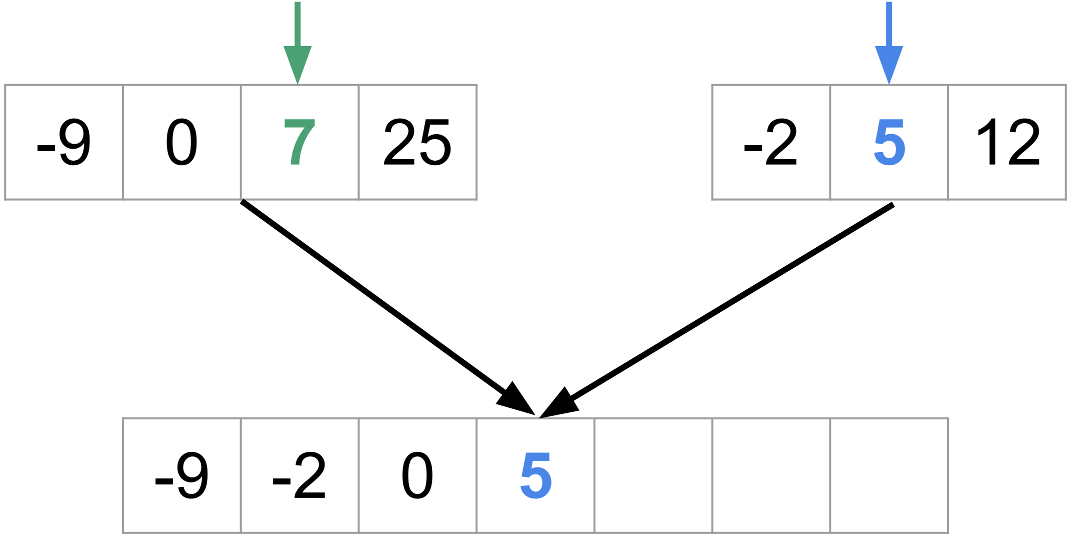

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-9	-2	0	5			

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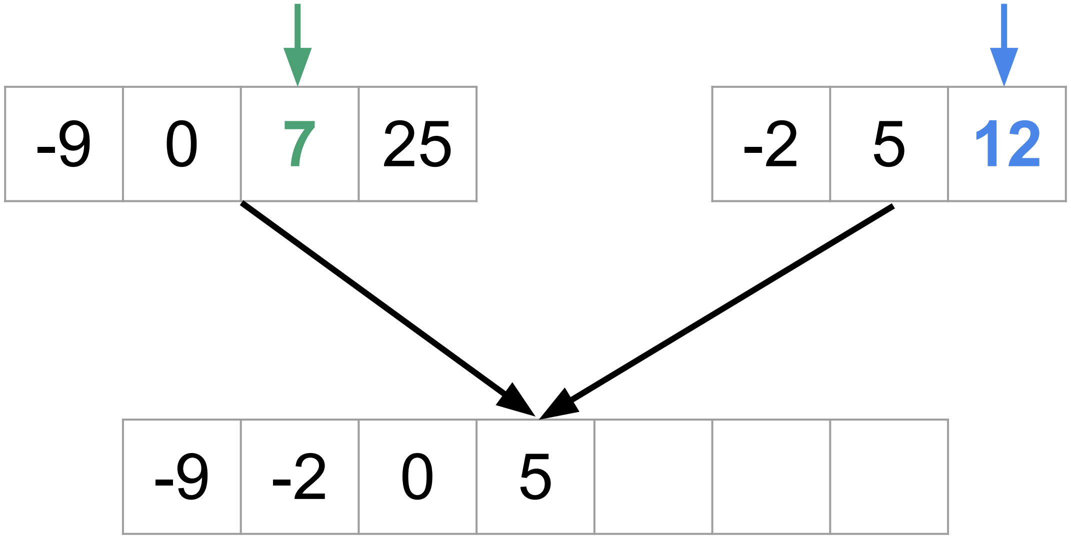

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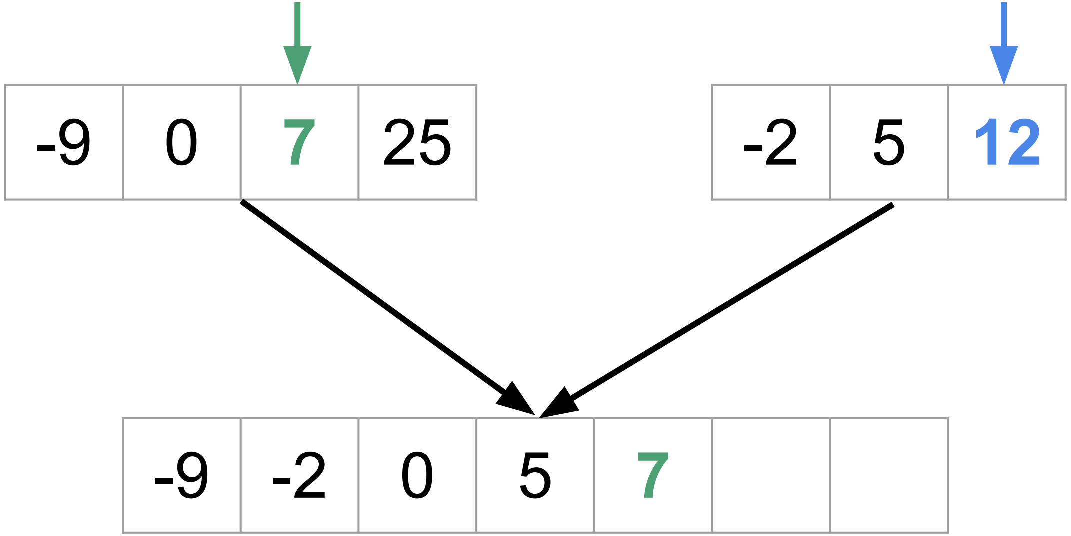

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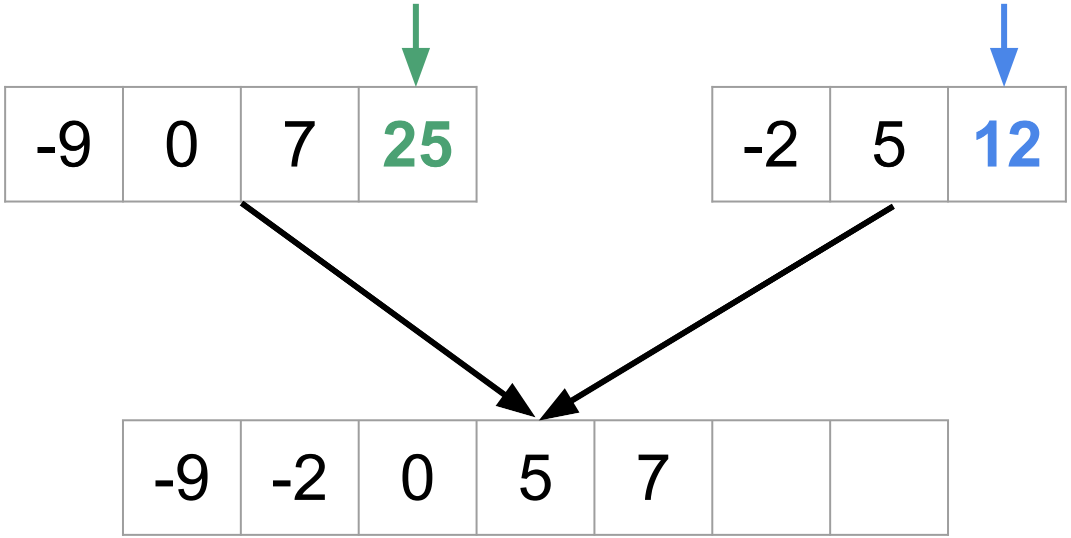

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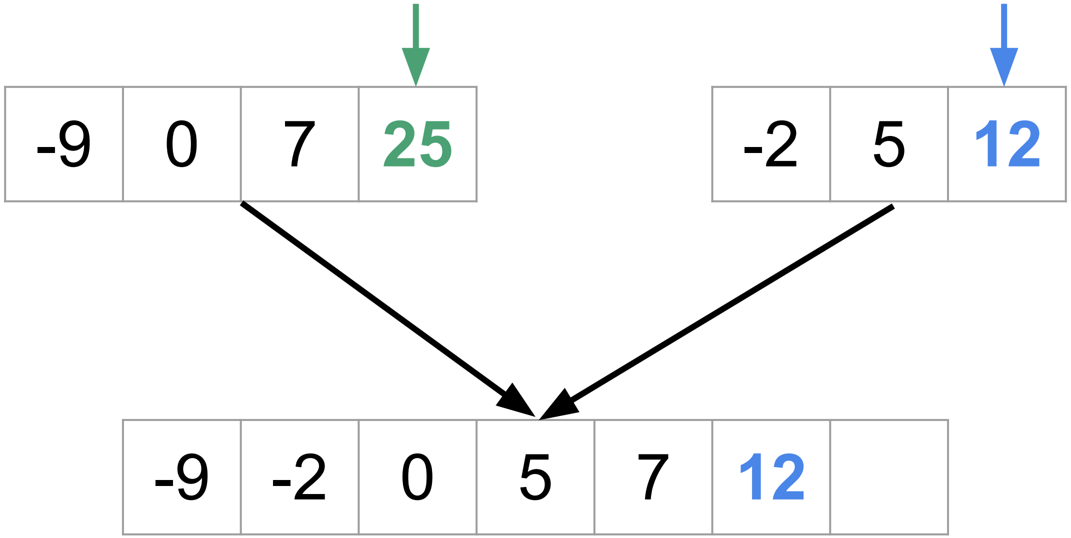

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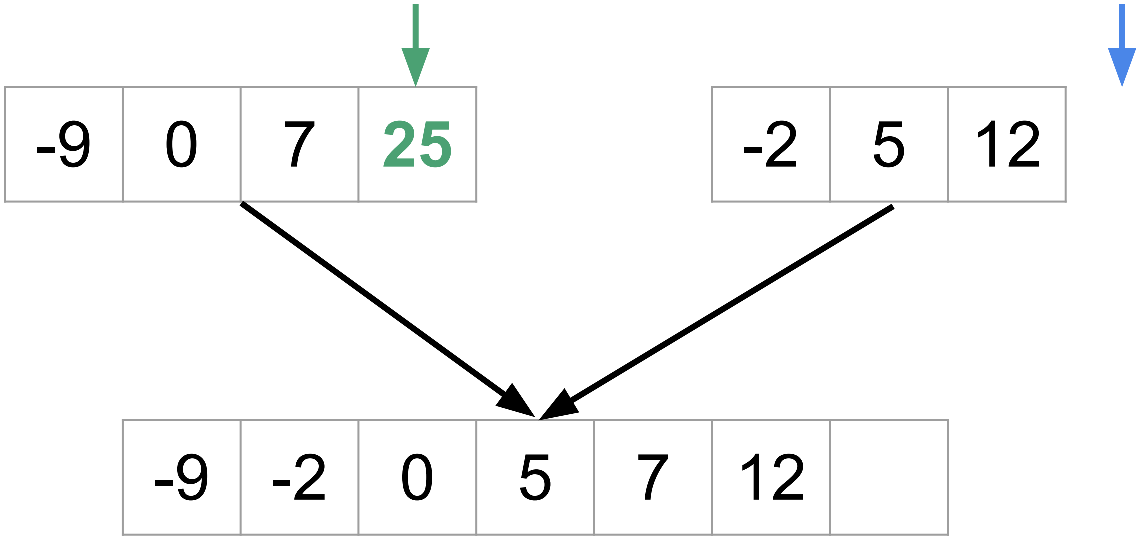

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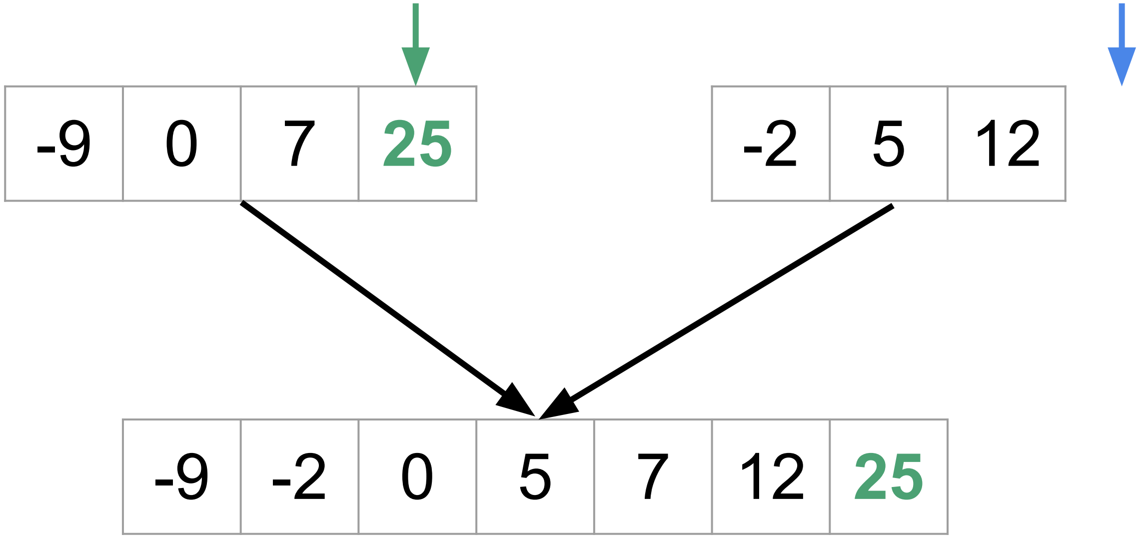

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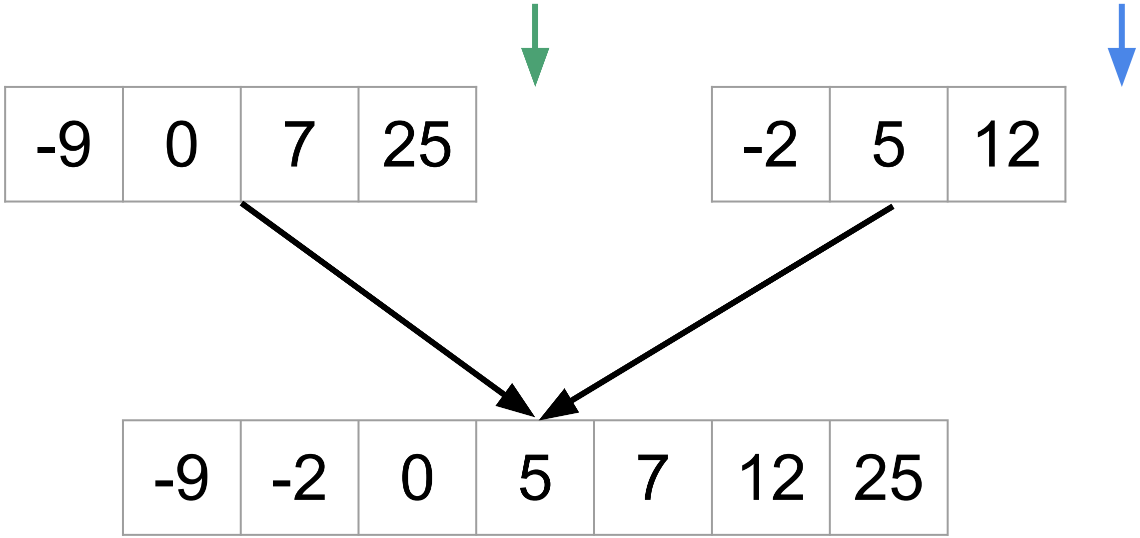

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Probably not!

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